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Quantised spin-1 field in flat Clifford–Klein space–times

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Abstract. We calculate the finite-temperature stress tensor for massive and massless spin-1 fields in static space–times $T \otimes M$, where M is a flat Clifford–Klein space form $\mathbb{R}^3/\Gamma$. The 2-torus and Klein bottle, $S^1 \otimes S^1 \otimes \mathbb{R}^1$ and $\mathbb{R}^1 \otimes K_2$ respectively, are chosen as M for detailed calculations, and we quote results for K_1.

1. Introduction

It is well known that the introduction of topological constraints upon a quantised field results in a shift of the stress–energy–momentum expectation values, this effect being named after Casimir (1948), who first calculated the vacuum energy between conducting plates associated with the electromagnetic field, and predicted an attractive force between the plates. More recently, Isham (1978) and Avis and Isham (1978) have discussed quantised fields in topologically non-trivial space–times, and DeWitt *et al* (1978) and Dowker and Banach (1978) have performed covariant Casimir calculations for the massless scalar field in several flat multiply connected spaces of the type $M \otimes T$, where

$$M = \mathbb{R}^3/\Gamma,$$

Γ being isomorphic to the fundamental group of M. Their method, in short, is to express the stress expectation values as the coincidence limit of a bilinear operator acting on the Feynman propagator for the manifold,

$$\langle T_{\mu\nu}(x)\rangle = \lim_{x' \to x} \vec{T}_{\mu\nu} \sum_{\gamma} \tilde{\Delta}_{\mathrm{F}}(x, x'\gamma)$$

where $\gamma \in \Gamma$ and $\tilde{\Delta}_{\mathrm{F}}(x, x')$ is the scalar Feynman propagator in $\mathbb{R}^3 \otimes T$. When γ is the identity element e, the infinite Minkowski stress is obtained, and each element $\gamma \neq e$ then contributes to the finite correction. Here, we shall extend this type of treatment to massive and massless spin-1 fields to discover their peculiar properties in non-trivial multiply connected spaces. It should be pointed out that Brown and Maclay (1969) have already performed rather elegant covariant Casimir calculations for the electromagnetic field between parallel plates, but the simplicity of this geometry allowed them to deal with the E and B fields without introducing vector potentials, and the above-mentioned peculiarities did not arise.

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2. Spin-1 formalism in $\mathbb{R}^3 \otimes T$

We first write down the Lagrangian density for a massive spin-1 field,

$$\mathscr{L}_V = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}m^2 A_\mu A^\mu$$

where $F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$ and $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. As $m \to 0$, the gauge breaking mass term vanishes, and in order to obtain a covariant propagator for the massless vector field, the Lagrangian must be modified. The physical consequences we understand in terms of the number of degrees of freedom associated with the field at each point in space. The massive vector field is constrained by

$$\partial_\mu A^\mu = 0$$

and hence has three degrees of freedom, that is, it has the helicity modes $\pm 1, 0$. In the massless limit, using $\partial_\mu A^\mu = 0$ to define the gauge, the zero-helicity modes decouple, as they could only couple to $\partial_\mu j^\mu$, which is zero. The massless field therefore has only two degrees of freedom. Foreseeing difficulties for $m \to 0$, we follow Lautrup's (1967) treatment of the electromagnetic field and then extend it to finite mass. He starts in the analogue of the classical Lorentz gauge, $\partial_\mu A^\mu = 0$, and adds a Lagrange multiplier to the usual massless Lagrangian density:

$$\mathscr{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \Lambda \partial_\mu A^\mu. \tag{1}$$

Λ is interpreted as a scalar field and we generalise equation (1) slightly to

$$\mathscr{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \Lambda \partial_\mu A^\mu + F(\Lambda(x)), \tag{2}$$

where $F(z)$ is a holomorphic function with $F(0) = F'(0) = 0$. The variation of equation (2) with respect to Λ gives

$$\partial_\mu A^\mu = F'(\Lambda(x)),$$

that is, the equation of motion for Λ gives us the constraint on A^μ. We now make the special choice

$$F(\Lambda) = \tfrac{1}{2}a\Lambda^2,$$

where a is a real number, and the gauge condition reads

$$\partial_\mu A^\mu = a\Lambda,$$

so we effectively have a gauge breaking term

$$\mathscr{L}_B = -(\partial_\mu A^\mu)^2/2a$$

in the Lagrangian. Now we restore the symmetry broken by $\mathscr{L}_B$ with a ghost field (e.g. Lee 1976), and then include the mass terms:

$$\mathscr{L}_G = -\partial_\mu \phi^* \partial^\mu \phi + m^2 \phi^* \phi.$$

Our final Lagrangian is

$$\mathscr{L} = \mathscr{L}_V + \mathscr{L}_B + \mathscr{L}_G = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}m^2 A_\mu A^\mu - (\partial_\mu A^\mu)^2/2a - \partial_\mu \phi^* \partial^\mu \phi + m^2 \phi^* \phi, \tag{3}$$

so that we now obtain the correct $m \to 0$ behaviour in the observables. The massive results are obtained by dropping any ghost contributions to the observables and taking the $a \to \infty$ limit which sets $\mathscr{L}_B$ to zero. Variation of equation (3) leads to the equations of motion

$$\{g_{\mu\nu}(\Box + m^2) + [(1-a)/a]\partial_\mu \partial_\nu\} A^\nu = 0 \tag{4}$$

and

$$(\Box + m^2)\phi = 0, \qquad (\Box + m^2)\phi^* = 0. \tag{5}$$

Equation (4) leads to the one-parameter family of Feynman propagators,

$$\langle 0|T\{A^\alpha(x)A^{\nu'}(x')\}|0\rangle = \mathrm{i}\tilde{D}^{\alpha\nu'}(x, x')$$
$$= -\mathrm{i}g^{\alpha\nu'}\tilde{\Delta}_F(x, x') + \frac{\mathrm{i}(1-a)}{(2\pi)^4}\int \frac{\mathrm{d}^4k \exp[-\mathrm{i}k_\mu(x-x')^\mu]k^\alpha k^\nu \delta_\nu^{\nu'}}{(k^2 - m^2 + \mathrm{i}\epsilon)(k^2 - m^2 a + \mathrm{i}\epsilon)} \tag{6}$$

where $\tilde{\Delta}_F$ is the scalar Feynman propagator satisfying

$$(\Box + m^2)\tilde{\Delta}_F(x, x') = -\delta(x - x'), \qquad \tilde{\Delta}_F(x, x') = \int \frac{\mathrm{d}^4k \exp[-\mathrm{i}k_\mu(x - x')^\mu]}{(2\pi)^4(k^2 - m^2 + \mathrm{i}\epsilon)}. \tag{7}$$

This family of gauges ($m \to 0$) is the quantum generalisation of the classical Lorentz gauge, in that

$$\partial_\mu \langle A^\mu \rangle = 0$$

for all physically realisable states. From equations (5) we have

$$\langle 0|T\{\phi(x)\phi^*(x')\}|0\rangle = \mathrm{i}\tilde{\Delta}_F(x, x').$$

Now, having expressions for the propagators, we require expressions for the stress expectation values in terms of these propagators. We define the stress tensor $T_{\mu\nu}$ as the response in the action functional to variations in the metric,

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}, \tag{8}$$

where $g = \det g_{\mu\nu}$, and

$$S = \int \mathrm{d}^4x \sqrt{-g}\mathscr{L}.$$

Replacing partial with covariant derivatives where necessary in equation (3), and using equation (8) we have, after returning to flat space,

$$T_{\mu\nu} = T^{\mathrm{V}}_{\mu\nu} + T^{\mathrm{B}}_{\mu\nu} + T^{\mathrm{G}}_{\mu\nu}$$

where

$$T^{\mathrm{V}}_{\mu\nu} = -F_{\mu\alpha}F_\nu{}^\alpha + \tfrac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + m^2 A_\mu A_\nu - \tfrac{1}{2}m^2 g_{\mu\nu}A_\alpha A^\alpha,$$
$$T^{\mathrm{B}}_{\mu\nu} = (1/a)(A_\mu \partial_\nu \partial_\alpha A^\alpha + A_\nu \partial_\mu \partial_\alpha A^\alpha - g_{\mu\nu}A_\alpha \partial^\alpha \partial_\beta A^\beta - \tfrac{1}{2}g_{\mu\nu}(\partial_\alpha A^\alpha)^2),$$
$$T^{\mathrm{G}}_{\mu\nu} = -\partial_\mu\phi\partial_\nu\phi^* - \partial_\nu\phi\partial_\mu\phi^* + g_{\mu\nu}(\partial_\alpha\phi\partial^\alpha\phi^* - m^2\phi\phi^*).$$

Note that $T^{\mathrm{G}}_{\mu\nu}$ corresponds to the minimally coupled scalar field, otherwise it would not cancel $\langle T_{\mu\nu}\rangle^{\mathrm{B}}$ as $m \to 0$. The stress vacuum expectation values are written as

$$\langle \tilde{T}_{\mu\nu}\rangle^{\mathrm{V}} = \mathrm{i}\lim_{x' \to x}[-g^{\sigma}_{\sigma'}(\partial_\mu \partial_{\nu'}\tilde{D}^{\sigma'}_{\sigma} - \partial_\mu \partial^{\sigma'}\tilde{D}_{\sigma\nu'} - \partial_\sigma \partial_{\nu'}\tilde{D}^{\sigma'}_{\mu} + \partial_\sigma \partial^{\sigma'}\tilde{D}_{\mu\nu'})$$
$$+ \tfrac{1}{2}g_{\mu\nu'}g^{\sigma}_{\sigma'}g^{\rho}_{\rho'}(\partial_\rho \partial^{\rho'}\tilde{D}^{\sigma'}_{\sigma} - \partial_\rho \partial^{\sigma'}\tilde{D}^{\rho'}_{\sigma}) + m^2\tilde{D}_{\mu\nu'} - g_{\mu\nu'}\tfrac{1}{2}m^2 g^{\rho}_{\rho'}\tilde{D}^{\rho'}_{\rho}], \tag{9a}$$

$$\langle \tilde{T}_{\mu\nu}\rangle^{\mathrm{B}} = (\mathrm{i}/a) \lim_{x'\to x}[-\tfrac{1}{2}g_{\mu\nu'}\partial_\beta\partial^{\alpha'}\tilde{D}^{\beta}_{\alpha'} - g_{\mu\nu'}g^{\rho'}_{\rho}\partial_{\rho'}\partial_{\sigma'}\tilde{D}^{\rho\sigma'} + \partial_{\nu'}\partial_{\rho'}\tilde{D}^{\rho'}_{\mu} + \partial_{\mu'}\partial_{\rho'}\tilde{D}^{\rho'}_{\nu}], \tag{9b}$$

$$\langle \tilde{T}_{\mu\nu}\rangle^{\mathrm{G}} = \mathrm{i} \lim_{x'\to x}[-\partial_\mu\partial_{\nu'} - \partial_{\nu'}\partial_\mu + g_{\mu\nu'}g^{\beta}_{\beta'}\partial_\beta\partial^{\beta'} - g_{\mu\nu'}m^2]\tilde{\Delta}_{\mathrm{F}}, \tag{9c}$$

where $(x-x')_\mu(x-x')^\mu > 0$. Up to now we have been working in Minkowski space and each $\langle \tilde{T}_{\mu\nu}\rangle$ is formally divergent. In the next section, we impose topological constraints on the spatial section of the manifold and take the $\langle T_{\mu\nu}\rangle$'s to be defined by the equations (9), only making the replacements $\tilde{\Delta}_{\mathrm{F}} \to \Delta_{\mathrm{F}}$ and $\tilde{D}_{\alpha\beta'} \to D_{\alpha\beta'}$, where Δ_{F} and $D_{\alpha\beta'}$ are the propagators for the multiply connected space minus the Minkowski contributions.

3. Topological constraints

We choose the subspace K_2 (Wolf 1967), the Klein bottle (non-orientable) which is the plane $\mathbb{R}^2$ factored by the group making the identities

$$\gamma: (x, y) \to (x + nb, (-1)^n y + 2kc),$$

where n, k are integers and the area of the Klein bottle is $b \times 2c$. For the ghost field, we interpret the identifications as

$$\phi(x, y, z, t) = \phi(x + nb, (-1)^n y + 2kc, z, t)$$

and for the vector field we write

$$A_\mu(x, y, z, t) = (1 - 2\delta_{y\mu})^n A_\mu(x + nb, (-1)^n y + 2kc, z, t) \tag{10}$$

with no sum on μ, that is, a reversal of vectors in the y direction after one traversal of the Klein bottle in the x direction. We shall work at finite temperature, which just involves introducing images along the imaginary time axis (e.g. Brown and Maclay 1969) in the propagators. So for our Klein bottle propagators we have

$$\Delta_{\mathrm{F}}(x, x') = \sum_{\gamma, r=-\infty}^{\infty}{}' \tilde{\Delta}_{\mathrm{F}}(\boldsymbol{x} - \boldsymbol{x}'\gamma, t - t' - \mathrm{i}r\beta),$$

where β is the inverse temperature. The Minkowski term $\gamma = e$, $r = 0$ is the source of divergences and hence is dropped in Σ'. We have

$$\Delta_{\mathrm{F}}(x, x') = \Delta'_{\mathrm{F}}(x, x') + \Delta''_{\mathrm{F}}(x, x'),$$

where

$$\Delta'_{F} = \sum_{\substack{n,k,r\\-\infty}}^{\infty}{}' \tilde{\Delta}_{\mathrm{F}}(x - x' - 2nb, y - y' - 2kc, z - z', t - t' - \mathrm{i}r\beta) \tag{11}$$

and

$$\Delta''_{\mathrm{F}} = \sum_{\substack{n,k,r\\-\infty}}^{\infty} \tilde{\Delta}_{\mathrm{F}}[x - x' - (2n+1)b, y + y' - 2kc, z - z', t - t' - \mathrm{i}r\beta]. \tag{12}$$

Likewise,

$$D_{\alpha\beta'}(x, x') = D'_{\alpha\beta'}(x, x') + D''_{\alpha\beta'}(x, x'),$$

where

$$D'_{\alpha\beta'} = \sum_{\substack{n,k,r\\-\infty}}^{\infty}{}' \tilde{D}_{\alpha\beta'}(x - x' - 2nb, y - y' - 2kc, z - z', t - t' - \mathrm{i}r\beta) \tag{13}$$

and

$$D''_{\alpha\beta'}=\sum_{\substack{n,k,r\\-\infty}}^{\infty} q_{\beta'}\tilde{D}_{\alpha\beta'}[x-x'-(2n+1)b,\, y+y'-2kc,\, z-z',\, t-t'-\mathrm{i}r\beta], \tag{14}$$

with $q_{\beta'}\equiv(1-2\delta_{y'\beta'})$, and no sum on β'. We notice that equations (11) and (13) are just the propagators for the spatial section $S^1\otimes S^1\otimes\mathbb{R}^1$, so the Klein bottle stress expectation values will consist of the 2-torus subspace results ($2b\times 2c$), with correction terms. Calculations are simplified by noting that

$$\begin{aligned}
&\partial_\mu\Delta'_{\mathrm{F}}=-\partial_{\mu'}\Delta'_{\mathrm{F}}\delta^{\mu'}_{\mu}, &&\partial_\mu\Delta''_{\mathrm{F}}=-q_\mu\partial_{\mu'}\Delta''_{\mathrm{F}}\delta^{\mu'}_{\mu},\\
&\partial_\mu D'_{\alpha\beta'}=-\partial_{\mu'}D'_{\alpha\beta'}\delta^{\mu'}_{\mu}, &&\partial_\mu D''_{\alpha\beta'}=-q_\mu\partial_{\mu'}D''_{\alpha\beta'}\delta^{\mu'}_{\mu}.
\end{aligned} \tag{15}$$

Also, if $\mu\neq\nu$ then

$$\lim_{x'\to x}\partial_\mu\partial_\nu\Delta_{\mathrm{F}}=\lim_{x'\to x}\partial_\mu\partial_\nu D^{\alpha\beta'}=g_{\mu\nu}=0, \tag{16}$$

and since elements of $\langle T_{\mu\nu}\rangle$ must be proportional to a combination of the above, all off-diagonal elements are zero.

4. The calculations

Incorporating equations (7), (15) and (16) into equation (9), we find, using the notation that $\langle T_{\mu\nu}\rangle'$ corresponds to the 2-torus part of our results, and $\langle T_{\mu\nu}\rangle''$ to the Klein bottle correction,

$$\begin{aligned}
\langle T_{\mu\nu}\rangle'^{\mathrm{V}}=\mathrm{i}\lim_{x'\to x}\Bigg[&\left(\frac{m^2}{2}g_{\mu\nu}-2\partial_\mu\partial_\nu\right)\Delta'_{\mathrm{F}}\\
&+\sum_{\omega}{}'\frac{m^2(1-a)}{2(2\pi)^4}\int\frac{\mathrm{d}^4k\,\exp[-\mathrm{i}k_\alpha(x-x'\omega)^\alpha](2k_\mu k_\nu-k^2g_{\mu\nu})}{(k^2-m^2+\mathrm{i}\epsilon)(k^2-m^2a+\mathrm{i}\epsilon)}\Bigg],\\
\langle T_{\mu\nu}\rangle'^{\mathrm{B}}=\frac{\mathrm{i}}{a}\lim_{x'\to x}\Bigg[&\left(-\frac{m^2}{2}g_{\mu\nu}-2\partial_\mu\partial_\nu\right)\Delta'_{\mathrm{F}}\\
&+\sum_{\omega}{}'\frac{(1-a)}{2(2\pi)^4}\int\frac{\mathrm{d}^4k\,\exp[-\mathrm{i}k_\alpha(x-x'\omega)^\alpha]k^2(g_{\mu\nu}k^2-4k_\mu k_\nu)}{(k^2-m^2+\mathrm{i}\epsilon)(k^2-m^2a+\mathrm{i}\epsilon)}\Bigg],
\end{aligned}$$

$$\langle T_{\mu\nu}\rangle'^{\mathrm{G}}=2\mathrm{i}\lim_{x'\to x}(\partial_\mu\partial_\nu\Delta'_{\mathrm{F}}),$$

where $\omega:(x,y,z,t)\to(x+2nb,\,y+2kc,\,z,\,t+\mathrm{i}r\beta)$.

$m=0$:

As required, we find

$$\langle T_{\mu\nu}\rangle_0'^{\mathrm{B}}+\langle T_{\mu\nu}\rangle_0'^{\mathrm{G}}=0$$

and so

$$\langle T_{\mu\nu}\rangle_0'=\langle T_{\mu\nu}\rangle_0'^{\mathrm{V}}=-2\mathrm{i}\lim_{x'\to x}\partial_\mu\partial_\nu\Delta'_{\mathrm{F}}. \tag{17}$$

$m \neq 0$:

Now

$$\langle T_{\mu\nu}\rangle_m'^{B} + \langle T_{\mu\nu}\rangle_m'^{G} \neq 0,$$

and to find the stress expectation values, we let $a \to \infty$, setting $\mathscr{L}_B$ to zero, and ignore any ghost contributions:

$$\langle T_{\mu\nu}\rangle_m' = \lim_{a\to\infty} (\langle T_{\mu\nu}\rangle'^{V} + \langle T_{\mu\nu}\rangle'^{B}) = i \lim_{\substack{a\to\infty \\ x'\to x}} \partial_\mu \partial_\nu\bigg(-(3+1/a)\Delta_F' + \sum_\omega{}' \frac{(1-1/a)\Box}{(2\pi)^4} \int \frac{d^4k \exp[-ik_\alpha(x-x'\omega)^\alpha]}{(k^2-m^2+i\epsilon)(k^2-m^2a+i\epsilon)}\bigg),$$

$$\langle T_{\mu\nu}\rangle_m' = -3i \lim_{x'\to x} \partial_\mu \partial_{\nu'} \Delta_F'. \tag{18}$$

Now we look at the Klein bottle correction terms:

$$\langle T_{\mu\nu}\rangle''^{V} = im^2 \lim_{x'\to x} \bigg(\frac{g_{\mu\nu}}{2}\Delta_F'' + \sum_\eta \frac{(1-a)}{(2\pi)^4} \int \frac{d^4k \exp[-ik_\alpha(x-x'\eta)^\alpha][k_\mu k_\nu q_\nu - (g_{\mu\nu}/2)(k^2+2k_y^2)]}{(k^2-m^2+i\epsilon)(k^2-m^2a+i\epsilon)}\bigg),$$

$$\langle T_{\mu\nu}\rangle''^{B} = \frac{i}{a} \lim_{x'\to x} \bigg[\Big(-\frac{g_{\mu\nu}}{2}m^2 + 2g_{\mu\nu}\partial_y\partial_y - 2\partial_\mu\partial_\nu q_\nu\Big)\Delta_F'' + \sum_\eta \frac{(1-a)}{(2\pi)^4} \int \frac{d^4k \exp[-ik_\alpha(x-x'\eta)^\alpha]k^2[2g_{\mu\nu}k_y^2 + (g_{\mu\nu}/2)k^2 - 2k_\mu k_\nu q_\nu]}{(k^2-m^2+i\epsilon)(k^2-m^2a+i\epsilon)}\bigg],$$

$$\langle T_{\mu\nu}\rangle''^{G} = -2i \lim_{x'\to x} (g_{\mu\nu}\partial_y\partial_y - \partial_\mu\partial_\nu q_\nu)\Delta_F'',$$

where $\eta: (x, y, z, t) \to [x+(2n+1)b, -y+2kc, z, t+ir\beta]$.

$m = 0$:

Again we have

$$\langle T_{\mu\nu}\rangle_0''^{B} + \langle T_{\mu\nu}\rangle_0''^{G} = 0,$$

so

$$\langle T_{\mu\nu}\rangle_0'' = \langle T_{\mu\nu}\rangle_0''^{V} = 0. \tag{19}$$

$m \neq 0$:

$$\langle T_{\mu\nu}\rangle_m'' = \lim_{a\to\infty} (\langle T_{\mu\nu}\rangle''^{V} + \langle T_{\mu\nu}\rangle''^{B})$$

$$= i \lim_{\substack{a\to\infty \\ x'\to x}} (g_{\mu\nu}\partial_y\partial_y - \partial_\mu\partial_\nu q_\nu)\bigg[\Big(1+\frac{1}{a}\Big)\Delta_F'' - \sum_\eta \frac{(1-1/a)\Box}{(2\pi)^4} \int \frac{d^4k \exp[-ik_\alpha(x-x'\eta)^\alpha]}{(k^2-m^2+i\epsilon)(k^2-m^2a+i\epsilon)}\bigg],$$

so

$$\langle T_{\mu\nu}\rangle''_m = \mathrm{i} \lim_{x'\to x} (g_{\mu\nu}\partial_y\partial_y - \partial_\mu\partial_\nu q_\nu)\Delta''_{\mathrm{F}}. \tag{20}$$

We could now calculate $\langle T_{\mu\nu}\rangle_0$ and $\langle T_{\mu\nu}\rangle_m$ explicitly in terms of the inverse temperature and the dimensions of the Klein bottle; however, our interest is the behaviour of the spin-1 field in non-trivial flat spaces, and so we shall express our results in terms of $\langle T_{\mu\nu}\rangle^{\mathrm{S}}$, the stress expectation values for a scalar field of the same mass, in the same space. Some scalar field results are quoted in the Appendix.

5. Results

We shall briefly cover the scalar field results. As in our case, the stress expectation values split up into a 2-torus contribution, $\langle T_{\mu\nu}\rangle'^{\mathrm{S}}$, and a Klein bottle correction, $\langle T_{\mu\nu}\rangle''^{\mathrm{S}}$. We may choose the equation of motion for the scalar field,

$$(\Box + m^2 + \xi R)\psi = 0,$$

where R is the scalar curvature of the manifold, to be either conformally or minimally coupled. When $\xi = 0$, the field is said to be minimally coupled, and when $\xi = \frac{1}{6}$, the field is conformally invariant as $m \to 0$. Since $T_{\mu\nu}$ depends upon the response of the action functional S to variations in the metric, and S depends upon the equation of motion for the field, then $T_{\mu\nu}$ depends upon our choice of ξ. Our notation is that $\langle T_{\mu\nu}\rangle^{\mathrm{S}}_{\mathrm{N}}$ corresponds to $\xi = 0$, and $\langle T_{\mu\nu}\rangle^{\mathrm{S}}_{\mathrm{C}}$ to $\xi = \frac{1}{6}$. It turns out that

$$\langle T_{\mu\nu}\rangle'^{\mathrm{S}}_{\mathrm{N}} = \langle T_{\mu\nu}\rangle'^{\mathrm{S}}_{\mathrm{C}} \equiv \langle T_{\mu\nu}\rangle'^{\mathrm{S}}$$

but

$$\langle T_{\mu\nu}\rangle''^{\mathrm{S}}_{\mathrm{N}} \neq \langle T_{\mu\nu}\rangle''^{\mathrm{S}}_{\mathrm{C}}.$$

Returning to the spin-1 field, we may express our results as

$$\langle T_{\mu\nu}\rangle'_0 = 2\langle T_{\mu\nu}\rangle'^{\mathrm{S}}, \tag{21}$$

$$\langle T_{\mu\nu}\rangle''_0 = 0(\neq \langle T_{\mu\nu}\rangle''^{\mathrm{S}}_{\mathrm{N}} \text{ or } \langle T_{\mu\nu}\rangle''^{\mathrm{S}}_{\mathrm{C}}), \tag{22}$$

$$\langle T_{\mu\nu}\rangle'_m = 3\langle T_{\mu\nu}\rangle'^{\mathrm{S}}, \tag{23}$$

$$\langle T_{\mu\nu}\rangle''_m = \langle T_{\mu\nu}\rangle''^{\mathrm{S}}_{\mathrm{N}}, \tag{24}$$

where of course we are relating vector and scalar fields of the same mass. Now equations (21) and (23) might have been expected from the arguments in § 2, where we discussed the number of degrees of freedom associated with the spin-1 field, and indeed we have a discontinuity in $\langle T_{\mu\nu}\rangle'$ at $m = 0$. However, equations (22) and (24) cannot be similarly explained, and result from global properties of the manifold. First we note that $\langle T_{\mu\nu}\rangle''_m$ is related to the minimally coupled scalar field value, and that its trace is given by

$$\langle T^\mu_\mu\rangle''_m = \mathrm{i}m^2 \lim_{x'\to x} q_\mu\left(g^\mu_\mu + \frac{1}{m^2}\partial_\mu\partial^\mu\right)\Delta''_{\mathrm{F}} = \mathrm{i}\lim_{x'\to x}(m^2 + 2\partial_y\partial_y)\Delta''_{\mathrm{F}}.$$

Hence, if $\langle T_{\mu\nu}\rangle''_0$ were given by $\langle T_{\mu\nu}\rangle''_0 = \lim_{m\to 0} k\langle T_{\mu\nu}\rangle''_m$, with k a non-zero real number, we would be presented with a trace anomaly, as $\langle T^\mu_\mu\rangle''_0$ would not vanish. However, $\langle T_{\mu\nu}\rangle''_0$ is not given by this limit and is trivially traceless, as seen in equation (22), reflecting the field's conformal invariance.

Mathematically, we may pinpoint the source of the rather strange behaviour exhibited in equation (24). If in the expression (14) for the vector field propagator we did not include the $q_{\beta'}$, that is, we did not reverse vectors in the y direction after a traversal of the Klein bottle in the x direction, we would calculate

$$\langle T_{\mu\nu}\rangle''_m = 3\langle T_{\mu\nu}\rangle''^{S}_{C},$$

which might have been explained in terms of our degrees of freedom argument, but our massless results could not be obtained from this treatment, since now

$$\langle T_{\mu\nu}\rangle''^{B}_{0} + \langle T_{\mu\nu}\rangle''^{G}_{0} \neq 0.$$

Omitting q^n_μ from equation (10) is, of course, not an alternative representation of the Klein bottle topological group. A consequence of equation (10) is that

$$\partial_\alpha\partial_\beta A_\mu(x, y, z, t) = q^n_\alpha q^n_\beta q^n_\mu \partial_\alpha\partial_\beta A_\mu(x+nb, (-1)^n y + 2kc, z, t)\dagger. \qquad (25)$$

If the A_μ's are dynamically constrained by equation (4) at some point (x', y', z', t'), then this condition, in conjunction with the topological constraint (25), should automatically lead to the A_μ's satisfying (4) at all image points of (x', y', z', t'). This is not the case if we omit q^n_μ from equation (25).

6. Electromagnetic field

Of particular interest is the $m = 0$ (electromagnetic) case. Since only $\langle T_{\mu\nu}\rangle''^{S}_{N}$ and $\langle T_{\mu\nu}\rangle''^{S}_{C}$ are y dependent, while $\langle T_{\mu\nu}\rangle'^{S}$ is not (see the Appendix), we find that the stress expectation values for the electromagnetic field, unlike those of the scalar field, are position independent. We cannot conclude that all observables are position independent. In particular, we have for the E and B fields

$$\begin{pmatrix}\langle E^2\rangle\\ \langle B^2\rangle\end{pmatrix} = -2\mathrm{i}\lim_{x'\to x}(\partial_0\partial_0\Delta'_F \pm \partial_y\partial_y\Delta''_F)$$

or

$$\begin{pmatrix}\langle E^2\rangle\\ \langle B^2\rangle\end{pmatrix} = -\frac{1}{\pi^2}\left(\sum_{n,k,r=-\infty}^{\infty}{}' \frac{4n^2b^2+4k^2c^2-3r^2\beta^2}{(4n^2b^2+4k^2c^2+r^2\beta^2)^3} \pm \sum_{n,k,r=-\infty}^{\infty} \frac{12(y-kc)^2-(2n+1)^2b^2-r^2\beta^2}{[(2n+1)^2b^2+4(y-kc)^2+r^2\beta^2]^3}\right).$$

Another interesting point is what we mean by demanding that the electromagnetic field satisfies the Klein bottle topological constraints. We have interpreted this as equation (10), which implies in terms of the physical fields that

$$F_{\mu\nu}(x, y, z, t) = q^n_\mu q^n_\nu F_{\mu\nu}(x+nb, (-1)^n y + 2kc, z, t)$$

or

$$\begin{bmatrix}E_x\\ E_z\\ B_y\end{bmatrix}(x, y, z, t) = \begin{bmatrix}E_x\\ E_z\\ B_y\end{bmatrix}(x+nb, (-1)^n y + 2kc, z, t)$$

† By this, we mean

$$\partial_\alpha\partial_\beta A_\mu(x, y, z, t)]_{(x,y,z,t)=(x',y',z',t')} = q^n_\alpha q^n_\beta q^n_\mu \partial_\alpha\partial_\beta A_\mu(x, y, z, t)]_{(x,y,z,t)=(x'+nb,(-1)^n y'+2kc,z',t')}.$$

and

$$\begin{bmatrix} E_y \\ B_x \\ B_z \end{bmatrix}(x, y, z, t) = (-1)^n \begin{bmatrix} E_y \\ B_x \\ B_z \end{bmatrix}(x+nb, (-1)^n y + 2kc, z, t).$$

Clearly, we may not replace A_μ by B_i in equation (10) to obtain equivalent conditions. What if instead of having equation (10) as our starting point, we put the Klein bottle constraints on both the physical fields, i.e.

$$\begin{bmatrix} E_i \\ B_i \end{bmatrix}(x, y, z, t) = q_i^n \begin{bmatrix} E_i \\ B_i \end{bmatrix}(x+nb, (-1)^n y + 2kc, z, t) \tag{26}$$

or

$$F_{\mu\nu}(x, y, z, t) = p_{\mu\nu}^n F_{\mu\nu}(x+nb, (-1)^n y + 2kc, z, t) \tag{27}$$

with no sum on μ, ν, where $p_{\mu\nu} = 1$ unless $\mu\nu = 02, 20, 13, 31$, in which case $p_{\mu\nu} = -1$? From equation (27) we have

$$\partial^\mu F_{\mu\nu}(x, y, z, t) = q_\mu^n p_{\mu\nu}^n \partial^\mu F_{\mu\nu}(x+nb, (-1)^n y + 2kc, z, t).$$

If $\partial^\mu F_{\mu\nu}(x, y, z, t) = 0$, it does not follow that $\partial^\mu F_{\mu\nu}(x+nb, (-1)^n y + 2kc, z, t) = 0$, that is, the E and B fields described by equation (26) do not globally satisfy Maxwell's equations. It is easy to show that neither are the remaining Maxwell equations globally satisfied. Noting however that (10) leads to a twisted B field, we may transfer the trivial representation of the Klein bottle group to the B field by twisting the A_μ and E fields, that is, including a factor of $(-1)^n$ on the right of (10) (and the corresponding ghost equation). This just changes the sign of $\langle T_{\mu\nu}\rangle_0''$, which is anyway zero.

7. Another multiply connected space

Here we quote the results for the space K_1 (Wolf 1967), where in the spatial section we make the point identifications

$$(x, y, z) \to (x+nb, (-1)^n y + 2kc, (-1)^n z).$$

Although this space, unlike $\mathbb{R}^1 \otimes K_2$, is orientable, it makes an interesting comparison with the Klein bottle in that now we have two perpendicular twists in the space, i.e. we must reverse vectors in both the y and z directions after a traversal in the x direction. This reduces to replacing equation (10) by

$$A_\mu(x, y, z, t) = (1 - 2\delta_{\mu y})^n (1 - 2\delta_{\mu z})^n A_\mu(x+nb, (-1)^n y + 2kc, (-1)^n z, t). \tag{28}$$

To quote the results, again we compare with the scalar field results in the same space. Once more we have a contribution to $\langle T_{\mu\nu}\rangle$ which is identical to the 2-torus $(2b \times 2c)$ result, and we will not repeat it. We give the correction contributions $\langle T_{\mu\nu}\rangle''$:

$$\begin{bmatrix} \langle T_{00}\rangle_0'' \\ \langle T_{xx}\rangle_0'' \end{bmatrix} = -2 \begin{bmatrix} \langle T_{00}\rangle_N''^S \\ \langle T_{xx}\rangle_N''^S \end{bmatrix}, \qquad \begin{bmatrix} \langle T_{yy}\rangle_0'' \\ \langle T_{zz}\rangle_0'' \\ \langle T_{zy}\rangle_0'' \end{bmatrix} = 2 \begin{bmatrix} \langle T_{yy}\rangle_N''^S \\ \langle T_{zz}\rangle_N''^S \\ \langle T_{zy}\rangle_N''^S \end{bmatrix} = 6 \begin{bmatrix} \langle T_{yy}\rangle_C''^S \\ \langle T_{zz}\rangle_C''^S \\ \langle T_{zy}\rangle_C''^S \end{bmatrix},$$

$$\begin{bmatrix} \langle T_{00}\rangle_m'' \\ \langle T_{xx}\rangle_m'' \end{bmatrix} = -1 \begin{bmatrix} \langle T_{00}\rangle_N''^S \\ \langle T_{xx}\rangle_N''^S \end{bmatrix}, \qquad \begin{bmatrix} \langle T_{yy}\rangle_m'' \\ \langle T_{zz}\rangle_m'' \\ \langle T_{zy}\rangle_m'' \end{bmatrix} = 3 \begin{bmatrix} \langle T_{yy}\rangle_N''^S \\ \langle T_{zz}\rangle_N''^S \\ \langle T_{zy}\rangle_N''^S \end{bmatrix} = 9 \begin{bmatrix} \langle T_{yy}\rangle_C''^S \\ \langle T_{zz}\rangle_C''^S \\ \langle T_{zy}\rangle_C''^S \end{bmatrix},$$

$\langle T^{\mu}_{\mu}\rangle''_0 = 0$ again, reflecting the conformal invariance of the massless spin-1 field. It is interesting to note that in the electromagnetic case, E_i or B_i may be substituted for A_μ in equation (28), resulting in equivalent constraints. This is a consequence of the orientability of K_1.

8. Conclusions

We already know that the global topology of the space in which a field is quantised can affect local quantities such as $\langle T_{\mu\nu}\rangle$. If we were to use the local 'degrees of freedom' argument to predict the spin-1 $\langle T_{\mu\nu}\rangle$ in terms of the scalar $\langle T_{\mu\nu}\rangle$, we would be assuming that the global topology imposes similar constraints on both scalar and vector functions. In certain spaces, such as Minkowski space and the 2-torus, this is true. However, in less trivial spaces, the scalar and vector constraints are quite distinguishable, resulting in $\langle T_{\mu\nu}\rangle$'s which are not simply related.

Future work will include an investigation of scalar and vector field thermodynamics in multiply connected spaces.

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Appendix: Some scalar field results

2-torus $(2b\times 2c)$:

$$\langle T_{00}\rangle'^{\mathrm{S}} = \frac{-m^2}{4\pi^2}\sum_{n,k,r \atop -\infty}^{\infty}{}' \left(\frac{\mathrm{K}_2(m\gamma_{nkr})}{\gamma^2_{nkr}} - mr^2\beta^2\frac{\mathrm{K}_3(m\gamma_{nkr})}{\gamma^3_{nkr}}\right),$$

$$\langle T_{00}\rangle'^{\mathrm{S}}_{m=0} = -\frac{1}{2\pi^2}\sum_{n,k,r \atop -\infty}^{\infty}{}' \left(\frac{4n^2b^2+4k^2c^2-3r^2\beta^2}{\gamma^6_{nkr}}\right),$$

where $\gamma_{nkr} \equiv (4n^2b^2+4k^2c^2+r^2\beta^2)^{1/2}$ and K_n is a modified Bessel function of the third kind.

$\mathbb{R}^1\otimes K_2$ *correction terms*:

$$\langle T_{00}\rangle''^{\mathrm{S}}_{\mathrm{N}} = -\frac{m^2}{4\pi^2}\sum_{n,k,r \atop -\infty}^{\infty}\left(\frac{2\mathrm{K}_2(m\sigma_{nkr})}{\sigma^2_{nkr}} - m[4(y-kc)^2+r^2\beta^2]\frac{\mathrm{K}_3(m\sigma_{nkr})}{\sigma^3_{nkr}}\right),$$

$$\langle T_{00}\rangle''^{\mathrm{S}}_{\mathrm{N}\atop m=0} = -\frac{1}{\pi^2}\sum_{n,k,r \atop -\infty}^{\infty}\left(\frac{(2n+1)^2b^2-4(y-kc)^2-r^2\beta^2}{\sigma^6_{nkr}}\right),$$

$$\langle T_{00}\rangle''^{\mathrm{S}}_{\mathrm{C}} = -\frac{m^2}{12\pi^2}\sum_{n,k,r \atop -\infty}^{\infty}\left(\frac{4\mathrm{K}_2(m\sigma_{nkr})}{\sigma^2_{nkr}} - m[4(y-kc)^2+3r^2\beta^2]\frac{\mathrm{K}_3(m\sigma_{nkr})}{\sigma^3_{nkr}}\right),$$

$$\langle T_{00}\rangle''^{S}_{\substack{C\\ m=0}} = -\frac{2}{3\pi^2}\sum_{n,k,r\atop -\infty}^{\infty}\left(\frac{(2n+1)^2b^2-2r^2\beta^2}{\sigma^6_{nkr}}\right),$$

where $\sigma_{nkr} \equiv [(2n+1)^2b^2+4(y-kc)^2+r^2\beta^2]^{1/2}$.

K_1 correction terms:

$$\langle T_{00}\rangle''^{S}_{N} = -\frac{m^2}{4\pi^2}\sum_{n,k,r\atop -\infty}^{\infty}\left(\frac{3\mathrm{K}_2(m\nu_{nkr})}{\nu^2_{nkr}} - m[4(y-kc)^2+4z^2+r^2\beta^2]\frac{\mathrm{K}_3(m\nu_{nkr})}{\nu^3_{nkr}}\right),$$

$$\langle T_{00}\rangle''^{S}_{\substack{N\\ m=0}} = -\frac{1}{2\pi^2}\sum_{n,k,r\atop -\infty}^{\infty}\left(\frac{3(2n+1)^2b^2-4(y-kc)^2-4z^2-r^2\beta^2}{\nu^6_{nkr}}\right),$$

$$\langle T_{00}\rangle''^{S}_{C} = -\frac{m^2}{12\pi^2}\sum_{n,k,r\atop -\infty}^{\infty}\left(\frac{5\mathrm{K}_2(m\nu_{nkr})}{\nu^2_{nk,r}} - m[4(y-kc)^2+4z^2+3r^2\beta^2]\frac{\mathrm{K}_3(m\nu_{nkr})}{\nu^3_{nkr}}\right),$$

$$\langle T_{00}\rangle''^{S}_{\substack{C\\ m=0}} = -\frac{1}{6\pi^2}\sum_{n,k,r\atop -\infty}^{\infty}\left(\frac{5(2n+1)^2b^2+4(y-kc)^2+4z^2-7r^2\beta^2}{\nu^6_{nkr}}\right),$$

where $\nu_{nkr} \equiv [(2n+1)^2b^2+4(y-kc)^2+4z^2+r^2\beta^2]^{1/2}$.

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